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KATEDRA
MATEMATIKY

KMA/APG1 Applications of Geometry 1

Review of Linear Algebra and Geometry

Outline

Linear Algebra

- Linear algebra studies vector spaces and linear mappings between them.
- The fundamental objects are:
  - vectors and operations on them (addition, scalar multiplication),
  - bases and coordinate systems (how vectors are represented),
  - matrices as a practical representation of linear mappings,
  - inner product (lengths, angles, orthogonality),
  - eigenvalues and eigenvectors (directions preserved by a transformation).
- Linear algebra forms a basic building block of mathematics and its applications.
- Many other disciplines are based on its concepts and methods, such as
  - geometry,
  - numerical mathematics,
  - optimization,
  - physics,
  - computer graphics,
  - machine learning.

Vector Space

- The basic algebraic object we will work with is a vector space.

Definition (Vector Space)

A vector space over the field $\mathbb{R}$ is a set V whose elements are called vectors, on which two operations are defined:

- vector addition: $+ : V \times V \to V$, denoted $\vec{u} + \vec{v}$ $\quad \big(\forall \vec{u}, \vec{v} \in V : \ \vec{u} + \vec{v} \in V\big)$,
- scalar multiplication: $\cdot : \mathbb{R} \times V \to V$, denoted $\alpha \vec{v}$ $\quad \big(\forall \vec{v} \in V, \ \forall \alpha \in \mathbb{R} : \ \alpha \vec{v} \in V\big)$,

such that for all $\vec{u}, \vec{v}, \vec{w} \in V$ and $\alpha, \beta \in \mathbb{R}$ the following axioms hold:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$,
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$,
3. There exists a zero vector $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$,
4. For every $\vec{v} \in V$ there exists $-\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = \vec{0}$,
5. $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$,
6. $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$,
7. $(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$,
8. $1\,\vec{v} = \vec{v}$.

Examples of Vector Spaces

- The following vector space plays a fundamental role:

Example (The vector space $\mathbb{R}^n$)

The vector space $\mathbb{R}^n$ consists of all n-tuples of real numbers:

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\},$$

with operations defined componentwise:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n),$$

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n).$$

Other examples of vector spaces:

- Spaces of real polynomials $\mathcal{P}_k$ of degree at most k.
- Spaces of all real matrices of a given size, e.g. $\mathbb{R}^{m\times n}$.
- Spaces of smooth functions on an interval, e.g. $C^1([a, b])$.

$$\vdots$$

Vector Subspace

Definition (Vector Subspace)

Let V be a vector space. A nonempty subset $U \subset V$ is called a vector subspace of V if for all $\vec{u}, \vec{v} \in U$ and all $\alpha, \beta \in \mathbb{R}$ we have

$$\alpha\vec{u} + \beta\vec{v} \in U.$$

- From the definition it follows that U contains the zero vector: for any $\vec{u}, \vec{v} \in U$ choosing $\alpha = \beta = 0$ gives
$$0\vec{u} + 0\vec{v} = \vec{0} \in U.$$
- Closure under addition is obtained by choosing $\alpha = \beta = 1$:
$$\vec{u} + \vec{v} \in U.$$
- Closure under scalar multiplication follows from $\alpha = \lambda$, $\beta = 0$:
$$\lambda\vec{u} \in U.$$
- Thus, a vector subspace is a subset of a vector space which is itself again a vector space with the same operations.

Example of a Vector Subspace in $\mathbb{R}^3$

Example (Plane through the origin)

Consider the set

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

- First, we verify that $U \neq \varnothing$, since the zero vector satisfies $0 + 0 + 0 = 0$.
- Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in U$ and $\alpha, \beta \in \mathbb{R}$; then any linear combination belongs to U:

$$\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2).$$

$$(\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2) = \alpha \underbrace{(x_1 + y_1 + z_1)}_{0} + \beta \underbrace{(x_2 + y_2 + z_2)}_{0} = 0.$$

- Hence, U is a vector subspace of $\mathbb{R}^3$ – geometrically, it is a plane passing through the origin.

- In general, the set of all solutions of a homogeneous linear system

$$\mathbf{A}\mathbf{x} = \mathbf{0}$$

is always a vector subspace of $\mathbb{R}^n$.

Basis and Coordinate System

- We will work only with finite-dimensional vector spaces, i.e., spaces that have a finite basis.

Definition (Basis and Dimension)

A basis of a (finite-dimensional) vector space V is an ordered finite set of vectors

$$\mathcal{B} = \{\vec{e}_1, \dots, \vec{e}_n\},$$

which is linearly independent and spans the entire space V. The number of basis vectors is called the dimension of V and is denoted by

$$\dim V = n.$$

Definition (Coordinate System)

Let $\mathcal{B} = \{\vec{e}_1, \dots, \vec{e}_n\}$ be a basis of V. Every vector $\vec{v} \in V$ can be uniquely expressed as

$$\vec{v} = \sum_{i=1}^{n} v_i \vec{e}_i,$$

where $v_1, \dots, v_n \in \mathbb{R}$ are the coordinates of the vector with respect to the basis $\mathcal{B}$.

- Analogously, we define bases, dimensions, and coordinates also for vector subspaces.

Identification with $\mathbb{R}^n$

Proposition (Identification with $\mathbb{R}^n$)

Every vector space of dimension n is isomorphic to $\mathbb{R}^n$ via the correspondence

$$\vec{v} = \sum_{i=1}^{n} v_i \vec{e}_i \quad \longleftrightarrow \quad (v_1, \ldots, v_n) \in \mathbb{R}^n.$$

- An isomorphism is a bijective linear mapping.
- Elements of $\mathbb{R}^n$ will be written in the form

$$\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n.$$

Example (Canonical basis)

The space $\mathbb{R}^n$ has the canonical basis

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \quad \ldots \quad \mathbf{e}_n = (0, 0, \ldots, 1).$$

- We will always work with the vector space $\mathbb{R}^n$ and use the canonical basis.

Inner Product in $\mathbb{R}^n$

- We consider $V = \mathbb{R}^n$ with vectors coordinates w.r.t. canonical basis.

Definition (Inner Product)

The inner product of two vectors $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

Definition (Norm of a Vector)

The norm (length) of a vector $\mathbf{u}$ is defined using the inner product:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{\sum_{i=1}^{n} u_i^2}.$$

- Vectors $\mathbf{u}, \mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.
- The angle $\varphi \in [0, \pi]$ between nonzero vectors is given by

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\varphi.$$

Example: Inner Product, Norm, and Angle

Example (Inner Product)

Consider the vectors

$$\mathbf{u} = (1, 2, 2), \qquad \mathbf{v} = (2, 0, 1).$$

- Inner product:

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 2 + 2 \cdot 0 + 2 \cdot 1 = 4.$$

- Norms of the vectors:

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3, \qquad \|\mathbf{v}\| = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}.$$

- Angle between the vectors:

$$\cos\varphi = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{4}{3\sqrt{5}}.$$

- It follows that the vectors $\mathbf{u}$ and $\mathbf{v}$ are neither orthogonal nor parallel.

Orthonormal Basis

Definition (Orthogonal basis of a subspace)

Let $U \subset \mathbb{R}^n$ be a vector subspace of dimension k. A basis

$$\mathcal{B} = \{\mathbf{d}_1, \ldots, \mathbf{d}_k\}$$

is called an orthogonal basis of the subspace U if for all $i \neq j$ we have

$$\mathbf{d}_i \cdot \mathbf{d}_j = 0.$$

Definition (Orthonormal basis of a subspace)

An orthogonal basis $\{\mathbf{d}_1, \ldots, \mathbf{d}_k\}$ is called orthonormal if, in addition, for all $i \in \{1, \ldots, k\}$ we have

$$\|\mathbf{d}_i\| = 1, \qquad \text{equivalently} \qquad \mathbf{d}_i \cdot \mathbf{d}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Proposition (Canonical basis in $\mathbb{R}^n$)

The canonical basis

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \quad \ldots \quad \mathbf{e}_n = (0, 0, \ldots, 1)$$

of $\mathbb{R}^n$ is an orthonormal basis.

Orthonormalization (Gram–Schmidt Process)

- Let $U \subset \mathbb{R}^n$ be a vector subspace and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis of U.
- Using the Gram–Schmidt process, we construct an orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ from this basis:
$$\mathbf{w}_1 := \mathbf{v}_1,$$
$$\mathbf{w}_i := \mathbf{v}_i - \sum_{j=1}^{i-1} \frac{\mathbf{v}_i \cdot \mathbf{w}_j}{\|\mathbf{w}_j\|^2}\, \mathbf{w}_j, \qquad i = 2, \ldots, k.$$
- From the orthogonal basis we obtain an orthonormal basis $\{\mathbf{d}_1, \ldots, \mathbf{d}_k\}$ of U by normalization:
$$\mathbf{d}_i := \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|}, \qquad i = 1, \ldots, k.$$

Example: Gram–Schmidt in $\mathbb{R}^3$

Example (Orthonormalizing a basis)

Consider the subspace $U \subset \mathbb{R}^3$ with basis

$$\mathbf{v}_1 = (1,1,0), \qquad \mathbf{v}_2 = (1,0,1).$$

- First, set
$$\mathbf{w}_1 = \mathbf{v}_1 = (1,1,0),$$
- Next, compute
$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\,\mathbf{w}_1 = (1,0,1) - \frac{1}{2}(1,1,0) = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 1\right) \sim (1,-1,2) = \tilde{\mathbf{w}}_2.$$
- Thus we have an orthogonal basis:
$$\mathbf{w}_1 = (1,1,0), \qquad \tilde{\mathbf{w}}_2 = (1,-1,2).$$
- By normalization we obtain an orthonormal basis:
$$\mathbf{d}_1 = \frac{1}{\sqrt{2}}(1,1,0), \qquad \mathbf{d}_2 = \frac{1}{\sqrt{6}}(1,-1,2).$$

Vector Projection

Proposition (Projection and decomposition)

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, where $\mathbf{v} \neq \mathbf{0}$. Then there exists a unique decomposition

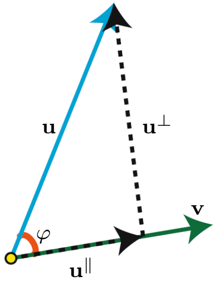
$$\mathbf{u} = \mathbf{u}^{\|} + \mathbf{u}^{\perp},$$

where $\mathbf{u}^{\|}$ is parallel to $\mathbf{v}$ and $\mathbf{u}^{\perp}$ is orthogonal to $\mathbf{v}$. The vector $\mathbf{u}^{\|}$ is called the projection of $\mathbf{u}$ onto the direction of $\mathbf{v}$, and

$$\mathbf{u}^{\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}.$$

Proof:

- We look for a vector of the form $\mathbf{u}^{\|} = \lambda \mathbf{v}$ such that
$$\mathbf{u}^{\perp} = \mathbf{u} - \mathbf{u}^{\|} \perp \mathbf{v}.$$
- The orthogonality condition:
$$(\mathbf{u} - \lambda \mathbf{v}) \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \lambda = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}.$$
- Substituting into $\mathbf{u}^{\|} = \lambda \mathbf{v}$ yields the desired formula.



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Example: Vector projection in $\mathbb{R}^3$

Example (Vector projection)

Consider the vectors

$$\mathbf{u} = (2, 1, 1), \qquad \mathbf{v} = (1, 1, 0).$$

- First compute the inner product and the squared norm of $\mathbf{v}$:

$$\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 3, \qquad \|\mathbf{v}\|^2 = 1^2 + 1^2 + 0^2 = 2.$$

- The projection of $\mathbf{u}$ onto the direction of $\mathbf{v}$ is

$$\mathbf{u}^{\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\, \mathbf{v} = \frac{3}{2}(1, 1, 0) = \left(\tfrac{3}{2}, \tfrac{3}{2}, 0\right).$$

- The orthogonal component is

$$\mathbf{u}^{\perp} = \mathbf{u} - \mathbf{u}^{\|} = (2, 1, 1) - \left(\tfrac{3}{2}, \tfrac{3}{2}, 0\right) = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 1\right).$$

- Verification of orthogonality:

$$\mathbf{u}^{\perp} \cdot \mathbf{v} = \tfrac{1}{2} \cdot 1 + \left(-\tfrac{1}{2}\right) \cdot 1 + 1 \cdot 0 = 0.$$

Projection onto a Subspace

- Analogously, we can define the projection of a vector onto a vector subspace U.
- For this, it is convenient to describe the subspace U by an orthogonal or orthonormal basis.
- Let $U \subset V$ be a vector subspace and let $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ be an orthogonal basis of U. Then the projection of $\mathbf{u}$ onto U is given by

$$\mathrm{proj}_U(\mathbf{u}) = \sum_{i=1}^{k} \frac{\mathbf{u} \cdot \mathbf{w}_i}{\|\mathbf{w}_i\|^2}\, \mathbf{w}_i.$$

- If $\{\mathbf{d}_1, \dots, \mathbf{d}_k\}$ is an orthonormal basis of U, we obtain the simplified form

$$\mathrm{proj}_U(\mathbf{u}) = \sum_{i=1}^{k} (\mathbf{u} \cdot \mathbf{d}_i)\, \mathbf{d}_i.$$

Example: Projection onto a subspace

Example (Projection onto a subspace)

Consider the subspace

$$U = \operatorname{span}\{\,\mathbf{w}_1, \mathbf{w}_2\,\} \subset \mathbb{R}^3, \qquad \mathbf{w}_1 = (1,1,0),\ \mathbf{w}_2 = (1,-1,0),$$

and the vector

$$\mathbf{u} = (2,1,3).$$

- The vectors $\mathbf{w}_1, \mathbf{w}_2$ are orthogonal:

$$\mathbf{w}_1 \cdot \mathbf{w}_2 = 1 - 1 + 0 = 0,$$

  so they form an orthogonal basis of the plane U.

- The projection of $\mathbf{u}$ onto U:

$$\operatorname{proj}_U(\mathbf{u}) = \frac{\mathbf{u}\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\,\mathbf{w}_1 + \frac{\mathbf{u}\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\,\mathbf{w}_2.$$

- Substituting:

$$\mathbf{u}\cdot\mathbf{w}_1 = 2+1+0 = 3, \qquad \mathbf{u}\cdot\mathbf{w}_2 = 2-1+0 = 1, \qquad \|\mathbf{w}_1\|^2 = \|\mathbf{w}_2\|^2 = 2.$$

$$\operatorname{proj}_U(\mathbf{u}) = \frac{3}{2}(1,1,0) + \frac{1}{2}(1,-1,0) = (2,1,0).$$

Cross Product in $\mathbb{R}^3$

Definition (Cross product)

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. The cross product is defined as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right).$$

- If the vectors $\mathbf{u}, \mathbf{v}$ are linearly dependent, then

$$\mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

- If the vectors $\mathbf{u}, \mathbf{v}$ are linearly independent, then:
  1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both vectors $\mathbf{u}$ and $\mathbf{v}$.
  2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin \varphi, \quad \varphi = \angle(\mathbf{u}, \mathbf{v})$.
  3. The direction of $\mathbf{u} \times \mathbf{v}$ is given by the right-hand rule.

Example: Cross product

Example (Cross product)

Consider the vectors $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (0, 1, 1)$.

- Cross product:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \left(\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right) = (-1, -1, 1).$$

- Orthogonality check:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (-1, -1, 1) \cdot (1, 0, 1) = -1 + 0 + 1 = 0,$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (-1, -1, 1) \cdot (0, 1, 1) = 0 - 1 + 1 = 0.$$

- Norms:

$$\|\mathbf{u}\| = \sqrt{2}, \qquad \|\mathbf{v}\| = \sqrt{2}, \qquad \|\mathbf{u} \times \mathbf{v}\| = \sqrt{3}.$$

- Angle between $\mathbf{u}$ and $\mathbf{v}$:

$$\cos\varphi = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{1}{2} \quad \Rightarrow \quad \sin\varphi = \frac{\sqrt{3}}{2}.$$

- Verification of the identity:

$$\|\mathbf{u}\| \, \|\mathbf{v}\| \, \sin\varphi = \sqrt{2} \cdot \sqrt{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{3} = \|\mathbf{u} \times \mathbf{v}\|.$$

Linear Mapping

Definition (Linear mapping)

Let V, W be vector spaces over $\mathbb{R}$. A mapping $\varphi : V \to W$ is called linear if for all $\vec{u}, \vec{v} \in V$ and all $\alpha, \beta \in \mathbb{R}$ we have

$$\varphi(\alpha\vec{u} + \beta\vec{v}) = \alpha\varphi(\vec{u}) + \beta\varphi(\vec{v}).$$

Proposition (Matrix form in $\mathbb{R}^n$)

Let $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, and let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then there exists a unique matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that

$$\varphi(\mathbf{u}) = \mathbf{A}\mathbf{u} \qquad \text{for all } \mathbf{u} \in \mathbb{R}^n.$$

- The columns of $\mathbf{A}$ are the images of the basis vectors of $\mathbb{R}^n$ under the mapping into $\mathbb{R}^m$:

$$\mathbf{A} = \begin{pmatrix} \varphi(\mathbf{e}_1) & \cdots & \varphi(\mathbf{e}_n) \end{pmatrix}.$$

- If $\varphi : V \to V$, then φ is called a linear transformation of V.
- In the case $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, the matrix $\mathbf{A}$ is a square $n \times n$ matrix.

Example of a linear transformation

Example (A linear transformation in $\mathbb{R}^2$)

Consider the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\varphi(x, y) = (2x + y,\ x - y).$$

- Matrix of the mapping in the canonical basis:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \varphi(\mathbf{u}) = \mathbf{A}\mathbf{u}.$$

- The columns of $\mathbf{A}$ are the images of the basis vectors:

$$\varphi(\mathbf{e}_1) = (2, 1), \qquad \varphi(\mathbf{e}_2) = (1, -1).$$

- For example, for the vector

$$\mathbf{u} = (1, 2)$$

we obtain

$$\varphi(\mathbf{u}) = \mathbf{A}\mathbf{u} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix},$$

that is,

$$\varphi(1, 2) = (4, -1).$$

Kernel and Image of a Linear Mapping

Definition (Kernel and image)

Let $\varphi : V \to W$ be a linear mapping.

- The kernel of φ is the set of all vectors mapped to the zero vector:

$$\ker \varphi = \{\vec{v} \in V \mid \varphi(\vec{v}) = \vec{0}\}.$$

- The image (range) of φ is the set of all values attained by the mapping:

$$\operatorname{Im} \varphi = \{\varphi(\vec{v}) \in W \mid \vec{v} \in V\}.$$

- The sets $\ker \varphi$ and $\operatorname{Im} \varphi$ are vector subspaces of V and W, respectively.

Proposition (Matrix form in $\mathbb{R}^n$)

Let $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, and let the linear mapping be given by $\varphi(\mathbf{u}) = \mathbf{A}\mathbf{u}$. Then

$$\ker \varphi = \ker \mathbf{A} = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{u} = \mathbf{0}\},$$

i.e., the subspace of all solutions of the homogeneous linear system $\mathbf{A}\mathbf{u} = \mathbf{0}$, and

$$\operatorname{Im} \varphi = \operatorname{Im} \mathbf{A} = \operatorname{span}\{\text{columns of } \mathbf{A}\},$$

i.e., the subspace spanned by the column vectors of $\mathbf{A}$.

Example: Computing the kernel

Example (Kernel of a linear mapping)

Consider the linear mapping $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \varphi(\mathbf{u}) = \mathbf{A}\mathbf{u}.$$

- We look for all vectors $\mathbf{u} = (x, y, z)$ satisfying

$$\mathbf{A}\mathbf{u} = \mathbf{0} \quad \iff \quad \begin{cases} x + y = 0, \\ y + z = 0. \end{cases}$$

- From the equations we obtain

$$x = -y, \qquad z = -y,$$

  hence

$$\mathbf{u} = \lambda(-1, 1, -1), \qquad \lambda \in \mathbb{R}.$$

- Therefore,

$$\ker \varphi = \operatorname{span}\{(-1, 1, -1)\},$$

  which is a line through the origin in $\mathbb{R}^3$.

Example: Image of a linear mapping

Example (Image of a linear mapping)

Consider the linear mapping $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ given by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

- The image is spanned by the columns of $\mathbf{A}$:

$$\operatorname{Im} \varphi = \operatorname{span}\{(1,0),(1,1),(0,1)\}.$$

- The third column is a linear combination of the first two:

$$(0,1) = (1,1) - (1,0),$$

  so it suffices to take

$$\operatorname{Im} \varphi = \operatorname{span}\{(1,0),(1,1)\}.$$

- These two vectors are linearly independent; hence they form a basis of $\mathbb{R}^2$, and therefore

$$\operatorname{Im} \varphi = \mathbb{R}^2.$$

- Therefore, the mapping is surjective.

Eigenvalues and Eigenvectors of a Linear Transformation

Definition (Eigenvalue and eigenvector)

Let $\varphi : V \to V$ be a linear transformation of a real vector space V. We say that a number $\lambda \in \mathbb{R}$ is an eigenvalue of φ and a nonzero vector $\vec{v} \in V$ is an eigenvector corresponding to λ if

$$\varphi(\vec{v}) = \lambda \vec{v}.$$

- For $V = \mathbb{R}^n$ and $\varphi(\mathbf{u}) = \mathbf{A}\mathbf{u}$ we obtain

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}.$$

Proposition (Characteristic equation)

Let $V = \mathbb{R}^n$ and $\varphi(\mathbf{u}) = \mathbf{A}\mathbf{u}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$. A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of φ if and only if

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

This polynomial is called the characteristic polynomial of the matrix $\mathbf{A}$.

- Even if the transformation is defined by a real matrix, its eigenvalues may be complex in general; therefore we work in the complexification of the space.
- For a fixed $\lambda \in \mathbb{C}$, the set of all solutions of

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0} \qquad \leftrightarrow \qquad \ker(\mathbf{A} - \lambda \mathbf{I})$$

is a vector subspace of $\mathbb{C}^n$, called the eigenspace corresponding to λ.

Example: Eigenvalues and eigenvectors

Example (Eigenvalues and eigenvectors)

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

- Eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3)^2 = 0 \quad \to \quad \lambda_1 = 1,\ \lambda_{2,3} = 3.$$

- Eigenspaces as kernels of the matrices $\mathbf{A} - \lambda\mathbf{I}$:
  - For $\lambda = 1$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \ker(\mathbf{A} - \mathbf{I}) = \operatorname{span}\{(1,-1,0)\}.$$

  - For $\lambda = 3$:

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \ker(\mathbf{A} - 3\mathbf{I}) = \operatorname{span}\{(1,1,0),(0,0,1)\}.$$

Outline

Affine and Euclidean Geometry

- In geometry we primarily work with points, not only with vectors.
- A vector space privileges the origin, but in geometry the origin has no special meaning. For example, every vector subspace passes through the origin.
- Therefore we introduce an affine space:
  - it allows us to work with points and vectors at the same time,
  - it has no preferred point – there is no natural origin,
  - thanks to its vector structure, we can use tools from linear algebra.
- Affine geometry describes:
  - points, lines, planes, etc.,
  - parallelism,
  - barycentric combinations of points,

  but it does not allow us to measure lengths or angles.
- To measure, we introduce a Euclidean space – an affine space equipped with an inner product.
- In Euclidean geometry we can work with:
  - distances,
  - angles and orthogonality,
  - lengths, areas, and volumes.

Affine Space

Definition (Affine space)

An affine space is a triple $(\mathbb{A}, V, \oplus)$, where:

- $\mathbb{A}$ is a set of points (the point set),
- V is a real vector space (the associated vector space),
- $\oplus : \mathbb{A} \times V \to \mathbb{A}$ is a map (the translation action),

satisfying:

1. $X \oplus \vec{0} = X$ for all $X \in \mathbb{A}$,
2. $X \oplus (\vec{u} + \vec{v}) = (X \oplus \vec{u}) \oplus \vec{v}$,
3. for every $X, Y \in \mathbb{A}$ there exists a unique $\vec{u} \in V$ such that $X \oplus \vec{u} = Y$.

- An affine space allows us to work with points and vectors simultaneously.
- The actual geometric constructions take place in the point set $\mathbb{A}$.
- Thanks to the associated vector space V, we can use tools from linear algebra in affine geometry.

Coordinate System

Definition (Affine frame)

Let $O \in \mathbb{A}_n$ and let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be a basis of the associated vector space V. The ordered $(n+1)$-tuple

$$\mathcal{R} = \{O, \vec{e}_1, \ldots, \vec{e}_n\}$$

is called an affine frame.

Definition (Coordinate system)

An affine frame $\mathcal{R}$ defines a map

$$\mathcal{S} : \mathbb{A}_n \to \mathbb{R}^n, \qquad X = O \oplus \sum_{i=1}^{n} x_i \vec{e}_i \longmapsto [x_1, \ldots, x_n],$$

which we call a coordinate system.

- The point O is called the origin of the coordinate system.

A model of an affine space in $\mathbb{R}^n$

Example (The affine space $\mathbb{R}^n$)

- After introducing coordinates, we work with the n-dimensional affine space over $\mathbb{R}$:

$$\mathbb{A}_n = \mathbb{R}^n, \quad V_n = \mathbb{R}^n.$$

- The translation is given by

$$[x_1, \dots, x_n] \oplus (u_1, \dots, u_n) = [x_1 + u_1, \dots, x_n + u_n].$$

- Two different copies of $\mathbb{R}^n$ appear:
  - as the set of points $\mathbb{A} = \mathbb{R}^n$,
  - as the vector space $V = \mathbb{R}^n$.
- To distinguish them, we write:
  - points with square brackets $\mathbf{x} = [x_1, \dots, x_n]$,
  - vectors with parentheses $\mathbf{u} = (u_1, \dots, u_n)$.
- Two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then uniquely determine the vector $\mathbf{u} = \mathbf{y} - \mathbf{x}$.

Affine Subspace

Definition (Affine subspace)

Let $(\mathbb{A}, V, \oplus)$ be an affine space, let $X \in \mathbb{A}$, and let $U \subset V$ be a vector subspace. The set

$$X \oplus U = \{ X \oplus \mathbf{u} \mid \mathbf{u} \in U \}$$

is called an affine subspace.

- Every affine subspace in $\mathbb{R}^n$ can be described as the solution set of a nonhomogeneous linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

- Unlike a vector subspace, an affine subspace does not generally pass through the origin.

Euclidean Space

Definition (Euclidean space)

By a Euclidean space $\mathbb{E}$ we mean an affine space whose associated vector space is equipped with an inner product.

- In what follows, we will always work with the model

$$(\mathbb{E}, V, \oplus) = (\mathbb{R}^n, \mathbb{R}^n, +),$$

and with the standard inner product on $\mathbb{R}^n$:

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

- Thanks to the inner product, in Euclidean space we can naturally work with distances and angles:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}, \qquad \cos \angle(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

Affine Mapping

Definition (Affine mapping)

Let $(\mathbb{A}_n, V, \oplus)$ and $(\mathbb{A}_m, W, \oplus')$ be affine spaces. A map $f : \mathbb{A}_n \to \mathbb{A}_m$ is called affine if there exists a linear map $\varphi : V \to W$ such that for all $X \in \mathbb{A}_n$ and $\vec{u} \in V$ we have

$$f(X \oplus \vec{u}) = f(X) \oplus' \varphi(\vec{u}).$$

Proposition (Matrix form of an affine mapping)

If $\mathbb{A}_n = \mathbb{R}^n$ and $\mathbb{A}_m = \mathbb{R}^m$ and $\varphi(\mathbf{u}) = \mathbf{A}\mathbf{u}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$, then an affine map has the form

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b},$$

where $\mathbf{b} \in \mathbb{R}^m$.

- An affine mapping preserves:
  - incidence,
  - parallelism,
  - the division ratio of points on a line,
  - barycentric combinations.
- But unlike a linear mapping, it does not map the origin to the origin.

Example of an affine transformation

Example (Linear part and translation)

Consider the affine transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- The map consists of a linear transformation

  $$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

  followed by a translation by the vector

  $$\mathbf{b} = (1, -1).$$

- For example, for the point

  $$\mathbf{x} = [2, 3]$$

  we obtain

  $$f(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

- In general, this affine transformation does not preserve lengths or angles.

Isometries

Definition (Isometry – affine isometry)

Let $\mathbb{E}_n$ and $\mathbb{E}_m$ be Euclidean spaces. An affine mapping $f : \mathbb{E}_n \to \mathbb{E}_m$ is called an isometry (affine isometry) if for all points $X, Y \in \mathbb{E}_n$ we have

$$\|f(X) - f(Y)\| = \|X - Y\|.$$

- The distance-preserving condition implies that necessarily $n = m$.

Proposition (Form of an isometry in $\mathbb{R}^n$)

If $\mathbb{E}_n = \mathbb{R}^n$, then a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry if and only if it has the form

$$f(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \mathbf{b},$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and $\mathbf{b} \in \mathbb{R}^n$.

- An isometry preserves:
  - lengths,
  - angles,
  - areas and volumes.

Types of isometries in $\mathbb{R}^2$ and $\mathbb{R}^3$

Isometries in $\mathbb{R}^2$:

- Identity.
- Translation by a vector $\mathbf{b}$.
- Rotation by an angle θ about a point P.
- Reflection across a line p.
- Glide reflection: reflection across a line p followed by a translation along p.

Isometries in $\mathbb{R}^3$:

- Identity.
- Translation by a vector $\mathbf{b}$.
- Rotation by an angle θ about an axis ℓ.
- Reflection across a plane π.
- Screw motion: a rotation about an axis ℓ combined with a translation along the axis ℓ.
- Glide reflection: reflection across a plane π combined with a translation within the plane π.
- Rotary reflection (roto-reflection): a rotation about an axis ℓ followed by a reflection across the plane perpendicular to ℓ.